

# Lie Groups and Quantum Mechanics

P.G.L. Leach<sup>\*</sup> and M.C. Nucci<sup>†</sup>

Dipartimento di Matematica e Informatica,  
Università di Perugia, 06123 Perugia, Italy

## Abstract

Mathematical modeling should present a consistent description of physical phenomena. We illustrate an inconsistency with two Hamiltonians – the standard Hamiltonian and an example found in Goldstein – for the simple harmonic oscillator and its quantisation. Both descriptions are rich in Lie point symmetries and so one can calculate many Jacobi Last Multipliers and therefore Lagrangians. The Last Multiplier provides the route to the resolution of this problem and indicates that the great debate about the quantisation of dissipative systems should never have occurred.

*Dedicated to the memory of Lev Berkovich*

## 1 Introduction

The mathematical description of Quantum Mechanics is largely due to the pioneering work of PAM Dirac who recognized the connection between the Hamiltonian description of Classical Mechanics and the operators he needed to describe the evolution of a quantal system<sup>1</sup> The essential idea was that one took the Hamiltonian and wrote it as an operator. Unfortunately the essence of the idea contained within is the seeds for confusion. In the case that one had an Hamiltonian of the form  $H = \frac{1}{2}p^2 + V(q)$  in the usual notation<sup>2</sup> there appeared to be no questions about the correctness of the

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<sup>\*</sup>permanent address: School of Mathematical Sciences, Westville Campus, University of KwaZulu-Natal, Durban 4000, Republic of South Africa, email: leachp@ukzn.ac.za, leachp@math.aegean.gr

<sup>†</sup>email: nucci@unipg.it

<sup>1</sup>Dirac commenced his career as an electrical engineer and it is perhaps not surprising that his recollection of the theory of Classical Mechanics was not quite perfect [4]. The story has it that he realised there was some connection on a Sunday afternoon, but did not have a useful text, such as Whittaker [13], at home to validate his memory and that it was necessary to wait until the following morning to access the University's library.

<sup>2</sup>One should note that Dirac referred to the energy which happened to be described by such a classical Hamiltonian. It is not evident if ever he contemplated quantisation using an Hamiltonian which did not represent the energy.

transition from the classical description to the quantal description. When the momenta and coordinates were not so conveniently separate as in the Hamiltonian above, it was necessary to devise some rules, such as normal ordering and the Weyl quantisation scheme, to deal with the essential noncommutativity of the operators. One of the beauties of Hamilton's description of mechanics is the invariance of his equations of motion under canonical transformation. In an important paper Leon van Hove [11] demonstrated that quantisation – by whatever rule one wanted to use – and canonical transformation did not necessarily commute. This poses a serious problem. If the description of a quantal problem is going to depend upon the choice of coordinate system, one is at least going to have to ensure that the coordinate system being used is physically correct.

In this paper we propose a procedure which obviates the constraint imposed by the conflict between consistent quantisation and the invariance of the Hamiltonian description under canonical transformation. We are motivated by a desire to maintain mathematical flexibility while at the same time being physically correct. It appears to us that the critical problem lies with the various quantisation schemes. Our proposal is to require that any quantisation scheme preserve the Noether point symmetries of the underlying Lagrangian. Indeed we go somewhat further. It is well known that there exist the potential for an unlimited number of Lagrangians for a given dynamical system. These Lagrangians can be constructed through the use of the Jacobi Last Multiplier and a knowledge of the Lie symmetries of the underlying Newtonian equation of motion. To each Lagrangian there corresponds an Hamiltonian so that in the very act of constructing the basis for the quantal problem one is already placed on the multiple horns of a dilemma. We propose that the Lagrangian of choice be that which possesses the maximal number of symmetries.

We illustrate our proposal with two representations of the Hamiltonian of the simple harmonic oscillator. In Section 2 we treat the simple harmonic oscillator in its standard representation. In Section 3 we use an alternative Hamiltonian to demonstrate what nonsense the usual quantisation schemes produce. In Section 4 and subsequently we show how we obtain a consistent description using the concepts mentioned above.

## 2 The Simple Harmonic Oscillator: Part I

The standard Hamiltonian for the simple harmonic oscillator is

$$H = \frac{1}{2} (p^2 + q^2) \tag{1}$$

and the corresponding Schrödinger equation is

$$i \frac{\partial u}{\partial t} = -\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} x^2 u. \tag{2}$$

Equation (2) possesses the Lie point symmetries

$$\Gamma_{1\pm} = \exp [\pm 2it] \{ \pm i\partial_t - x\partial_x \pm [x^2 + \frac{1}{2}] u\partial_u \} \quad (3)$$

$$\Gamma_3 = i\partial_t \quad (4)$$

$$\Gamma_{4\pm} = \exp [\pm it] \{ \pm \partial_x - xu\partial_u \} \quad (5)$$

$$\Gamma_6 = u\partial_u \quad (6)$$

$$\Gamma_7 = s(t, x)\partial_u, \quad (7)$$

where  $s(t, x)$  is any solution of (2).

The algebra of the Lie point symmetries is  $\{sl(2, R) \oplus_s W\} \oplus_s \infty A_1$ , where  $W$  is the three-element Heisenberg-Weyl algebra.

We use  $\Gamma_{4\pm}$  to construct a similarity solution of (2). The associated Lagrange's system is

$$\frac{dt}{0} = \frac{dx}{\pm 1} = \frac{du}{-xu} \quad (8)$$

from which it is evident that the characteristics are  $t$  and  $u \exp [\pm \frac{1}{2}x^2]$ . To obtain a solution which has the correct behaviour at  $\pm\infty$  we choose the characteristic with the positive sign in the exponential and set

$$u(t, x) = g(t) \exp [-\frac{1}{2}x^2], \quad (9)$$

where  $g(t)$  is determined by the substitution of (9) into (2). It is a simple calculation to show that

$$g(t) = \exp [-\frac{1}{2}it] \quad (10)$$

up to a multiplicative constant which we ignore. Consequently we have the ground-state solution

$$u_0(t, x) = \exp [-\frac{1}{2}it - \frac{1}{2}x^2] \quad (11)$$

corresponding to the symmetry  $\Gamma_{4+}$ .

Since (11) is a solution of (2), we may use it in  $\Gamma_7$ . Then

$$\begin{aligned} [\Gamma_{4-}, \Gamma_7]_{LB} &= [\exp [-it] \{ -\partial_x - xu\partial_u \}, \exp [-\frac{1}{2}it - \frac{1}{2}x^2] \partial_u]_{LB} \\ &= 2x \exp [-\frac{3}{2}it - \frac{1}{2}x^2] \partial_u \end{aligned}$$

and so we have obtained another solution, namely

$$u_1(t, x) = 2x \exp [-\frac{3}{2}it - \frac{1}{2}x^2]. \quad (12)$$

Further solutions are constructed in a similar fashion.

The symmetry,  $\Gamma_3$ , acts as an eigenvalue operator since

$$\begin{aligned} \Gamma_3 u_0 &= \frac{1}{2} u_0 \\ \Gamma_3 u_1 &= \frac{3}{2} u_1 \end{aligned}$$

*etc.*

### 3 The Simple Harmonic Oscillator: Part II

One of the attractive features of Hamiltonian Mechanics is the preservation of the structure of Hamilton's equations of motion under canonical transformation. In his well-known text the unfortunately late Herbert Goldstein presents an alternative Hamiltonian for the simple harmonic oscillator as [2] [ex 18, p 433]

$$H = \frac{1}{2} \left( \frac{1}{q^2} + p^2 q^4 \right). \quad (13)$$

The canonical transformation between (1) and (13) is

$$\tilde{q} = -\frac{1}{q} \quad \tilde{p} = pq^2. \quad (14)$$

We have a choice of methods to obtain the Schrödinger equation corresponding to the Hamiltonian (13).

If we use the normal-ordering method [1], [7] for the product involving the two canonical variables, the Schrödinger equation is

$$2i \frac{\partial u}{\partial t} = -x^4 \frac{\partial^2 u}{\partial x^2} - 4x^3 \frac{\partial u}{\partial x} + \left( \frac{1}{x^2} - 6x^2 \right) u. \quad (15)$$

Equation (15) possesses the Lie point symmetries

$$\begin{aligned} \Phi_{1\pm} &= \exp [\pm 2it] \left\{ \pm i \partial_t + x \partial_x + \left[ -\frac{1}{2} \pm \frac{1}{x^2} \right] u \partial_u \right\} \\ \Phi_3 &= \partial_t \\ \Phi_4 &= u \partial_u \\ \Phi_5 &= s(t, x) \partial_u, \end{aligned} \quad (16)$$

where  $s(t, x)$  is a solution of (15).

If we use the Weyl quantisation scheme [12], we obtain

$$2i \frac{\partial u}{\partial t} = -x^4 \frac{\partial^2 u}{\partial x^2} - 4x^3 \frac{\partial u}{\partial x} + \left( \frac{1}{x^2} - 3x^2 \right) u \quad (17)$$

and the Lie point symmetries are

$$\begin{aligned} \Sigma_{1\pm} &= \exp [\pm 2it] \left\{ \pm i \partial_t + x \partial_x + \left[ -\frac{1}{2} \pm \frac{1}{x^2} \right] u \partial_u \right\} \\ \Sigma_3 &= \partial_t \\ \Sigma_4 &= u \partial_u \\ \Sigma_5 &= s(t, x) \partial_u, \end{aligned} \quad (18)$$

where now  $s(t, x)$  is a solution of (17).

Thirdly, if we use the method proposed in [8], the Schrödinger equation is

$$2i \frac{\partial u}{\partial t} = -x^4 \frac{\partial^2 u}{\partial x^2} - 4x^3 \frac{\partial u}{\partial x} + \left( \frac{1}{x^2} - 2x^2 \right) u \quad (19)$$

which has the Lie point symmetries

$$\begin{aligned} \Delta_{1\pm} &= \exp[\pm 2it] \left\{ \pm i \partial_t + x \partial_x + \left[ -\frac{1}{2} \pm \frac{1}{x^2} \right] u \partial_u \right\} \\ \Delta_3 &= i \partial_t \\ \Delta_{4\pm} &= \exp[\pm it] \left\{ x^2 \partial_x + \left[ -x \pm \frac{1}{x} \right] u \partial_u \right\} \\ \Delta_6 &= u \partial_u \\ \Delta_7 &= s(t, x) \partial_u \end{aligned} \quad (20)$$

and now  $s(t, x)$  is a solution of (19).

There does seem to be something of a divergence!

In principle we can use the symmetries in (16), (18) and (20) to construct solutions of the respective Schrödinger equations and obtain the eigenvalues just as we did for the Schrödinger equation (2). In the case of (19) we obtain

$$u_0(t, x) = x^{-1} \exp \left[ -\frac{1}{2} it - \frac{1}{2x^2} \right] \quad E_0 = \frac{1}{2}. \quad (21)$$

The results for (15) and (17) are impossible being, respectively,

$$\exp \left[ \frac{1}{2} it - \frac{1}{2x^2} \right] \left\{ A (xe^{it})^{(-1-i\sqrt{15}/2)} + B (xe^{it})^{(-1+i\sqrt{15}/2)} \right\}$$

and

$$\exp \left[ \frac{1}{2} it - \frac{1}{2x^2} \right] \left\{ A (xe^{it})^{(-1-i\sqrt{3}/2)} + B (xe^{it})^{(-1+i\sqrt{3}/2)} \right\},$$

where  $A$  and  $B$  are constants of integration. Neither the normal ordering method nor the Weyl quantisation procedure leads to a result which is physical!

## 4 The Last Multiplier of Jacobi

Jacobi's Last Multiplier is a solution of the linear partial differential equation [3, 13],

$$\sum_{i=1}^n \frac{\partial(Ma_i)}{\partial x_i} = 0. \quad (22)$$

The relationship between the Jacobi Last Multiplier and the Lagrangian, *videlicet*

$$\frac{\partial^2 L}{\partial \dot{x}^2} = M \quad (23)$$

for a one-degree-of-freedom system, is perhaps not widely known.

If two multipliers,  $M_1$  and  $M_2$ , are known, their ratio is a first integral.

In the case of a conservative system with the standard energy integral

$$E = \frac{1}{2}\dot{x}^2 + V(x) \quad (24)$$

and Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - V(x) \quad (25)$$

it is evident from (23) that one multiplier is a constant – taken to be 1 without loss of generality – and so all multipliers are first integrals. This combined with (23) is a simple recipe for the generation of a Lagrangian. One has

$$\frac{\partial^2 L}{\partial \dot{x}^2} = 1 \implies L = \frac{1}{2}\dot{x}^2 + \dot{x}f_1(t, x) + f_2(t, x), \quad (26)$$

where  $f_1$  and  $f_2$  are arbitrary functions of integration. Naturally different multipliers give rise to different Lagrangians.

Lagrange's equation of motion for (26) is

$$\ddot{x} + \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = 0 \quad (27)$$

whereas that for (25) is

$$\ddot{x} + V'(x) = 0. \quad (28)$$

The requirement that the two Newtonian equations be the same is

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = V'(x). \quad (29)$$

This constraint may be expressed through setting

$$f_1 = \frac{\partial g}{\partial x}, \quad f_2 = \frac{\partial g}{\partial t} - V(x), \quad (30)$$

where  $g(t, x)$  is an arbitrary function of its arguments. Consequently the Lagrangian, (26), becomes

$$L = \frac{1}{2}\dot{x}^2 - V(x) + \dot{x}\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} = \frac{1}{2}\dot{x}^2 - V(x) + \dot{g}, \quad (31)$$

ie, the functions  $f_1$  and  $f_2$  are a consequence of the arbitrariness of a Lagrangian with respect to a total time derivative, the gauge function.

## 5 Algebraic Consistency of Gauge-Variant Lagrangians [10]

The canonical momentum for (26) is

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} + f_1 \quad (32)$$

so that

$$H = p\dot{x} - L = \frac{1}{2}p^2 - pf_1 + \frac{1}{2}f_1^2 - f_2 \quad (33)$$

is the Hamiltonian. Whether one uses the Weyl quantisation formula or the symmetrisation of  $pf_1$  makes no difference to the form of the Schrödinger Equation corresponding to (33) which is

$$2i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + 2if_1\frac{\partial u}{\partial x} + \left(f_1^2 - 2f_2 + i\frac{\partial f_1}{\partial x}\right)u. \quad (34)$$

The Schrödinger Equation, (34), is quite general. We now introduce the simple harmonic oscillator with Newtonian equation of motion  $\ddot{x} + k^2x = 0$  so that the constraint (29) is

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial x} = k^2x. \quad (35)$$

The Lie point symmetries of (34) subject to the constraint (35) are

$$\begin{aligned} \Gamma_1 &= \cos(kt) \partial_x + [\cos(kt)f_1 - \sin(kt)kx] iu \partial_u \\ \Gamma_2 &= -\sin(kt) \partial_x - [\sin(kt)f_1 + \cos(kt)kx] iu \partial_u \\ \Gamma_3 &= \partial_t + \left(f_2 + \frac{1}{2}k^2x^2\right) iu \partial_u \\ \Gamma_4 &= \cos(2kt) \partial_t - \sin(2kt)kx \partial_x \\ &\quad + \left[ i \cos(2kt) \left(f_2 - \frac{1}{2}k^2x^2\right) - k \sin(2kt) \left(ixf_1 - \frac{1}{2}\right) \right] u \partial_u \\ \Gamma_5 &= -\sin(2kt) \partial_t - \cos(2kt)kx \partial_x \\ &\quad - \left[ i \sin(2kt) \left(f_2 - \frac{1}{2}k^2x^2\right) + k \cos(2kt) \left(ixf_1 - \frac{1}{2}\right) \right] u \partial_u \\ \Gamma_6 &= u \partial_u \\ \Gamma_7 &= s(t, x) \partial_u, \end{aligned} \quad (36)$$

where  $s(t, x)$  is a solution of (34), which is a representation of the well-known algebra,  $\{sl(2, R) \oplus_s W\} \oplus_s \infty A_1$ , of the Schrödinger Equation for the one-dimensional linear oscillator and related systems. The presence of the functions  $f_1$  and  $f_2$  subject to the constraint (35) does not affect the number of Lie point symmetries of (34) vis-à-vis the number for the Schrödinger Equation for the simple harmonic oscillator.

## 6 Creation and Annihilation Operators

We write  $\Gamma_1$  and  $\Gamma_2$  and  $\Gamma_4$  and  $\Gamma_5$  as

$$\Gamma_{1\pm} = \exp[\pm kit] \{ \partial_x + i(f_1 \pm ikx) u \partial_u \} \quad (37)$$

$$\Gamma_{4\pm} = \exp[\pm 2kit] \{ \partial_t \pm kix \partial_x + i \left[ (f_2 - \frac{1}{2}k^2x^2) \pm k (ixf_1 - \frac{1}{2}) \right] u \partial_u \}. \quad (38)$$

The normal route to the solution of the Schrödinger Equation, (34), is to use the symmetries (37) which are the time-dependent progenitors of the well-known creation and annihilation operators of Dirac in the case that  $f_1$  and  $f_2$  are restricted as above.

To solve the Schrödinger Equation, (34), using Lie's method we reduce (34) to an ordinary differential equation by using the invariants of the symmetries as a source of the variables. We must also be cognisant of the need for the solution of (34) to satisfy the boundary conditions at  $\pm\infty$ .

With this requirement in mind we take  $\Gamma_{1+}$ . The associated Lagrange's system is

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{i(f_1 + kix)u} \quad (39)$$

which gives the characteristics  $t$  and  $u \exp \left[ \frac{1}{2}kx^2 - ig(t, x) \right]$ , where we have made use of the first of (30) and the fact that  $t$  is a characteristic. To find the solution corresponding to  $\Gamma_{1+}$  we write

$$u(t, x) = h(t) \exp \left[ -\frac{1}{2}kx^2 + ig(t, x) \right], \quad (40)$$

where  $h(t)$  is to be determined, and substitute it into (34) which simplifies to

$$i\dot{h} = \frac{1}{2}kh$$

so that

$$h(t) = \exp \left[ -\frac{1}{2}kit \right]$$

and

$$u(t, x) = \exp \left[ -\frac{1}{2}kit - \frac{1}{2}kx^2 + ig(t, x) \right]. \quad (41)$$

With  $g = 0$  we recognise the ground-state solution for the time-dependent Schrödinger Equation of the simple harmonic oscillator.

We use  $\Gamma_{1-}$  as a time-dependent 'creation operator'. If we write the left hand side of (41) as  $u_0$ , we can have a solution symmetry of the form

$$\Gamma_{70} = u_0(t, x)\partial_u, \quad (42)$$

where the subscript, 7j, means that we are using the symmetry  $\Gamma_7$  with the specific solution,  $u_j(t, x)$ . We use the closure of the Lie algebra under the operation of taking the Lie Bracket to obtain further solutions. Thus

$$[\Gamma_{1-}, \Gamma_{70}]_{LB} = \left\{ -kx + i\frac{\partial g}{\partial x} - (if_1 + kx) \right\} \exp \left[ -\frac{3}{2}kit - \frac{1}{2}kx^2 + ig \right] \partial_u \quad (43)$$

so that we have

$$u_1(t, x) = -2kx \exp \left[ -\frac{3}{2}kit - \frac{1}{2}kx^2 + ig \right]. \quad (44)$$

Likewise  $[\Gamma_{1-}, \Gamma_{71}]_{LB}$  gives

$$u_2(t, x) = (4k^2x^2 - 2k) \exp \left[ -\frac{5}{2}kit - \frac{1}{2}kx^2 + ig \right]. \quad (45)$$

$\Gamma_{4\pm}$  act as double annihilation and creation operators.

Finally the Lie Bracket of  $i\Gamma_3$  with  $\Gamma_7$  yields the energy. For example with  $\Gamma_{72}$  one has

$$[i\Gamma_3, \Gamma_{72}]_{LB} = \frac{5}{2}ku_2\partial_u. \quad (46)$$



## 7 A Proliferation of Lagrangians [9]

Lie's method [5, 6] for the calculation of the Jacobi Last Multiplier is firstly to find the value of

$$\Delta = \det \begin{bmatrix} e_{ij} \\ s_{ij} \end{bmatrix}, \quad (47)$$

in which the matrix is square with the elements  $e_{ij}$  being the vector field of the set of first-order differential equations by which the system is described and the elements,  $s_{ij}$ , being the coefficient functions of the number of symmetries of the given system necessary to make the matrix square. If  $\Delta$  is not zero, the corresponding multiplier is  $M = \Delta^{-1}$ .

We use the simple harmonic oscillator with equation of motion

$$\ddot{q} + k^2 q = 0 \quad (48)$$

as our vehicle.

To determine Jacobi's Last Multipliers one writes the system as a set of first-order ordinary differential equations and (48) becomes

$$\begin{aligned} \dot{u}_1 &= u_2 \\ \dot{u}_2 &= -k^2 u_1 \end{aligned} \quad (49)$$

with associated vector field

$$X_{SHO} = \partial_t + u_2 \partial_{u_1} - k^2 u_1 \partial_{u_2}. \quad (50)$$

As a linear second-order ordinary differential equation (48) possesses eight Lie point symmetries. In terms of the variables used in (49) the eight vectors are

$$\begin{aligned} \Gamma_1 &= \cos kt \partial_{u_1} - k \sin kt \partial_{u_2} \\ \Gamma_2 &= \sin kt \partial_{u_1} + k \cos kt \partial_{u_2} \\ \Gamma_3 &= u_1 \partial_{u_1} + u_2 \partial_{u_2} \\ \Gamma_4 &= \partial_t \\ \Gamma_5 &= \cos 2kt \partial_t - k u_1 \sin 2kt \partial_{u_1} - (2k^2 u_1 \cos 2kt - k u_2 \sin 2kt) \partial_{u_2} \\ \Gamma_6 &= \sin 2kt \partial_t + k u_1 \cos 2kt \partial_{u_1} - (2k^2 u_1 \sin 2kt + k u_2 \cos 2kt) \partial_{u_2} \\ \Gamma_7 &= u_1 \cos kt \partial_t - k u_1^2 \sin kt \partial_{u_1} - (k^2 u_1^2 \cos kt + k u_1 u_2 \cos kt + u_2^2 \cos kt) \partial_{u_2} \\ \Gamma_8 &= u_1 \sin kt \partial_t + k u_1^2 \cos kt \partial_{u_1} - (k^2 u_1^2 \sin kt - k u_1 u_2 \cos kt + u_2^2 \sin kt) \partial_{u_2}. \end{aligned} \quad (51)$$

Since the vector field, (50), has three elements, two symmetries of the eight in (51) are required for the computation of the determinant. There are twenty-eight possibilities. Of these fourteen are zero. Of the fourteen nonzero determinants there are really only three distinct possibilities. The other multipliers can be expressed as combinations of these three. Consequently we list only the three basic multipliers plus

a single combination of obvious interest. They are

$$\begin{aligned}
JLM_{12} &= k \\
JLM_{13} &= \frac{1}{ku_1 \sin kt + u_2 \cos kt} \\
JLM_{23} &= \frac{1}{-ku_1 \cos kt + u_2 \sin kt} \\
JLM_{34} &= [JLM_{13}^{-2} + JLM_{23}^{-2}]^{-1} = \frac{1}{u_2^2 + k^2 u_1^2}.
\end{aligned} \tag{52}$$

For each of these four multipliers we can calculate a Lagrangian and we list them with the constraint imposed on the two functions of integration,  $f_1(t, u_1)$  and  $f_2(t, u_1)$ , after the Lagrangian to which it applies.

$$L_{12} = \frac{1}{2}u_2^2 + f_1 u_2 + f_2,$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = k^2 u_1;$$

$$\begin{aligned}
L_{13} &= \sec^2 kt [\log(ku_1 \sin kt + u_2 \cos kt) (ku_1 \sin kt + u_2 \cos kt) \\
&\quad - u_2 \cos kt - ku_1 \sin kt] + f_1 u_2 + f_2,
\end{aligned}$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$\begin{aligned}
L_{23} &= \operatorname{cosec}^2 kt [\log(-ku_1 \cos kt + u_2 \sin kt) (-ku_1 \cos kt + u_2 \sin kt) \\
&\quad - u_2 \sin kt + ku_1 \cos kt] + f_1 u_2 + f_2,
\end{aligned}$$

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0;$$

$$L_{34} = \frac{u_2}{ku_1} \arctan\left(\frac{u_2}{ku_1}\right) - \frac{1}{2} \log\left(\frac{u_2^2}{k^2 u_1^2} + 1\right) + f_1 u_2 + f_2,$$

$$u_1 \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = 1.$$

The number of Noether point symmetries associated with these Lagrangians varies [9]. There are five for  $L_{12}$ , three for  $L_{13}$  and  $L_{23}$  and two for  $L_{34}$ , i.e.

$$\begin{aligned}
L_{12} &\implies \Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5, \Gamma_6 \\
L_{13} &\implies \Gamma_1, \Gamma_4 + \Gamma_5, -k\Gamma_3 + \Gamma_6 \\
L_{23} &\implies \Gamma_2, -\Gamma_4 + \Gamma_5, k\Gamma_3 + \Gamma_6 \\
L_{34} &\implies \Gamma_3, \Gamma_4
\end{aligned} \tag{53}$$

$L_{12}$  is the only Lagrangian with five Noether point symmetries. Not one of the fourteen Lagrangians has four, an additional three have three, seven more have two and there are none with one or zero Noether point symmetries [9].

For each of these Lagrangians one may construct an Hamiltonian. They are, with the relationship between the momentum,  $p$ , and  $u_2$ ,

$$\begin{aligned} u_2 &= p - f_1 \\ H_{12} &= \frac{1}{2}p^2 - pf_1 + \frac{1}{2}f_1^2 - f_2 \end{aligned}$$

$$\begin{aligned} u_2 &= -ku_1 \tan(kt) + \frac{\exp[\cos(kt)(p - f_1)]}{\cos(kt)} \\ H_{13} &= \frac{\exp[\cos(kt)(p - f_1)]}{\cos^2(kt)} - ku_1 \tan(kt)(p - f_1) - f_2 \end{aligned}$$

$$\begin{aligned} u_2 &= ku_1 \cot(kt) + \frac{\exp[\sin(kt)(p - f_1)]}{\sin(kt)} \\ H_{23} &= \frac{\exp[\sin(kt)(p - f_1)]}{\sin^2(kt)} + ku_1 \cot(kt)(p - f_1) - f_2 \end{aligned}$$

$$\begin{aligned} u_2 &= ku_1 \tan[ku_1(p - f_1)] \\ H_{34} &= \frac{1}{2} \log[\tan^2[ku_1(p - f_1)] + 1] - f_2 \end{aligned}$$

with the constraints on  $f_1$  and  $f_2$  listed above. Apart from  $H_{12}$  the determination of the corresponding Schrödinger Equation is a nontrivial exercise.

## 8 Conclusion

If one accepts that a mathematical description of a physical reality should be consistent with the Physics, it is quite evident from the elementary examples we have considered here that the standard approaches to the quantisation of a Classical Hamiltonian System are fraught with the possibility of error. In the case of different representations of the one-dimensional simple harmonic oscillator we have been able to present a consistent approach for quantisation. This is because of the generous supply of Lie point symmetries with which this problem is endowed. We have seen that in addition to some of the standard examples presented in the general literature that it is possible to construct Lagrangians and hence Hamiltonians of a far greater number than one would normally expect, indeed normally desire. In our analysis of the Noether point symmetries of these Lagrangians we saw that the number of symmetries varied from two to five. Given the close connection of the Noether symmetries to the Lie symmetries of the corresponding Schrödinger equation the question of the correct choice of a Lagrangian

can be quite important in the search for closed-form solutions. In addition we have the further question of an appropriate method for quantisation. By using the approach which we advocated in [8] we were able to obtain the correct result for the Hamiltonian, (13). It has been known for over fifty years [11] that quantisation and nonlinear canonical transformations have no guarantee of consistency. We argued then that there should be a preservation of the algebraic structure. *A fortiori* with the plethora of Lagrangians for the standard representation of the simple harmonic oscillator and the considerable variation in the number of Noether symmetries the need for the preservation of the algebraic structure becomes even more evident.

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